§ 2 Quick Review

2.1 Set Notations

Set : A well-defined collection of distinct objects (elements)

$\subseteq$: subset

$\in$: belongs to

$|A|$: Number of elements (Cardinality) of the set A

Example 2.1.1

$S = \{1, 2, 3\}$

That means S is a set containing 3 elements, namely 1, 2 and 3.

OR : $1, 2, 3 \in S$

If $T = \{1, 2, 3, 4\}$, then we say S is a subset of T, or $S \subseteq T$.

That means every element in S is also an element in T.

Notations often used :

$\mathbb{N}$: set of all natural numbers (positive integers)

$\mathbb{Z}$: set of all integers

$\mathbb{Q}$: set of all rational numbers

$\mathbb{R}$: set of all real numbers

$\mathbb{C}$: set of all complex numbers

ϕ : empty set, i.e. $\phi = \{\ \}$ Nothing

$[a,b]$: set of all real numbers x such that $a \le x \le b$

(a,b) : set of all real numbers x such that $a < x < b$

$[a,\infty)$: set of all real numbers x such that $a \le x$

Example 2.1.2

- $\phi \subseteq A$ for any set A.
- $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$
- Let $A = \{\{1\}, \{2\}, \{1,2\}\}$. A consists of 3 elements, but in fact each element is again a set

Example 2.1.3

Set of all positive even integers

$= \{2, 4, 6, \ldots\}$

$= \{2m : m \in \mathbb{N}\}$

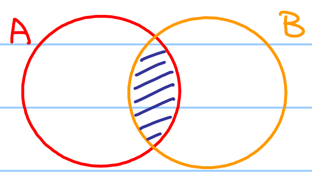
i.e this set consists of elements of the form $2m$ such that $m \in \mathbb{N}$.

Set of all positive odd integers = ? (How to describe ?)

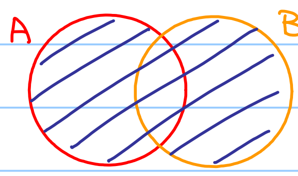
Answer : $\{2m-1 : m \in \mathbb{N}\}$

Set Operations :

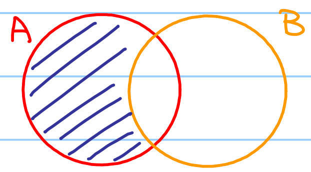
Let A, B be two sets.



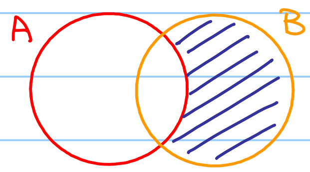
Intersection : $A \cap B$



Union : $A \cup B$



Relative complement of B in A : $A \setminus B$



Relative complement of A in B : $B \setminus A$

Example 2.1.4

Let $A = \{1, 2\}$, $B = \{2, 3\}$, $C = \{3\}$

- $A \cap B = \{2\}$ $\quad A \cap C = \phi$
- $A \cup B = A \cup C = \{1, 2, 3\}$

(Sometimes, we use $A \amalg C$ instead of $A \cup C$ to emphasize it is a disjoint union, i.e. $A \cap C = \phi$.)

- $A \setminus B = \{1\}$ $\quad B \setminus A = \{3\}$

Example 2.1.5

$\mathbb{R} \setminus \{2\}$: set of all real numbers except 2

(Caution : We cannot write $\mathbb{R} \setminus 2$ as 2 is not a set !)

$A \times B$: Product of two sets A and B defined by $\{(a,b) : a \in A \text{ and } b \in B\}$

Example 2.1.6

- Let $A = \{1,2,3\}$, $B = \{4,5\}$.

  $A \times B = \{(1,4),(1,5),(2,4),(2,5),(3,4),(3,5)\}$

  $B \times A = \{(4,1),(4,2),(4,3),(5,1),(5,2),(5,3)\}$

- $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2 = \{(x,y) : x,y \in \mathbb{R}\}$ = set of points on a plane

$\forall$: for all

$\exists$: there exists (at least one)

$\exists!$: there exists unique

$\rightarrow$: if ..., then $\Rightarrow$: implies

$\leftrightarrow$: if and only if $\Leftrightarrow$: equivalent to

s.t. : such that

$\wedge$: and $\vee$: or

$\neg$: not

Example 2.1.7

$\forall\, y \in (0,\infty)$, $\exists\, x \in \mathbb{R}$ s.t. $x^2 = y$

$\downarrow$ translate

For all positive real numbers y, there exists (at least one) real number x such that $x^2 = y$.

(In fact, $x = \sqrt{y}$ or $x = -\sqrt{y}$)

$\forall\, y \in (0,\infty)$, $\exists!\, x \in (0,\infty)$ s.t. $x^2 = y$

$\downarrow$ translate

For all positive real numbers y, there exists unique positive real number x such that $x^2 = y$.

(In fact, $x = \sqrt{y}$ only!)

Example 2.1.8

Let $x>0$, $\quad y=\sqrt{x} \xrightarrow{\checkmark} y^2=x \qquad\qquad y=\sqrt{x} \Rightarrow y^2=x$

$\qquad y^2=x \xrightarrow{\times} y=\sqrt{x}$ (why?)

Example 2.1.9

In $\triangle ABC$,

$\angle ABC=90^\circ \Rightarrow AB^2+BC^2=AC^2$ (Pyth. thm.)

$AB^2+BC^2=AC^2 \Rightarrow \angle ABC=90^\circ$ (Converse of Pyth. thm.)

we denote it by $\angle ABC=90^\circ \Leftrightarrow AB^2+BC^2=AC^2$

2.2 Relation

Definition 2.2.1

A relation R from a set A to a set B is a subset R of $A\times B$.

Also, we say that "a is related to b" if $(a,b)\in R$, sometimes it can be denoted by aRb or $a\sim b$.

We denote the relation by R or $\sim$.

In particular, if $A=B$, then R is said to be a relation defined on A.

Example 2.2.1

Let $A=\{2,3\}$, $B=\{3,4,5,6\}$.

Let R be a relation from A to B given by $R=\{(a,b)\in A\times B \cdot b \text{ is divisible by } a\}$

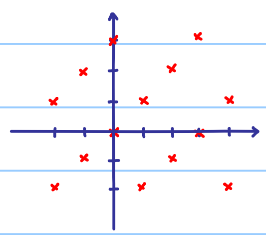
Then $R=\{(2,4),(2,6),(3,3),(3,6)\}$

Remark: Given a relation R from a set A to a set B, R consists of pairs of (a,b) such that a and b are related in some sense.

Example 2.2.2

Let R be a relation defined on $\mathbb{Z}$ which is given by $(a,b)\in R$ if $b-a$ is divisible by 3.

Then the relation can be visualized as:



Example 2.2.3

Let $\mathcal{L}$ be the set of all straight lines in $\mathbb{R}^2$ (in usual sense)

Let R be a relation defined on $\mathcal{L}$ which is given by $(l_1, l_2) \in R$ if $l_1 // l_2$.

Example 2.2.4

Let $\mathbb{P}$ be the set of line segment in $\mathbb{R}^2$ (in usual sense)

and let $\varphi : \mathbb{P} \to \mathbb{R}^+$ be the distance function, i.e $\varphi(s)$ = distance of line segment s.

Define $\cong$ as a relation on $\mathbb{P}$ such that $s_1 \cong s_2$ if $\varphi(s_1) = \varphi(s_2)$.

We can also define another relation $\leq$ on $\mathbb{P}/\cong$ such that $[s_1] \leq [s_2]$ if $\varphi(s_1) \leq \varphi(s_2)$.

Example 2.2.5

Let " | " be a relation on $\mathbb{N}$ such that $m, n \in \mathbb{N}$ and $n | m$ if m is divisible by n.

Definition 2.2.2

Let $\sim$ be a relation defined on a set A.

Then $\sim$ is said to be an equivalence relation on A if

1) (reflexive) $a \sim a$ for all $a \in A$

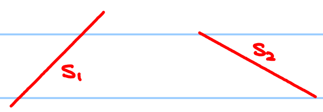
2) (symmetric) if $a \sim b$, then $b \sim a$

3) (transitive) if $a \sim b$ and $b \sim c$, then $a \sim c$.

Example 2.2.6 / Exercise 2.2.1

The relations in example 2.2.2, 2.2.3 and $\cong$ in example 2.2.4 are actually equivalence relations

Remark: An equivalence relation on a set A tells us whether two elements in A are equal in some sense.



$\varphi(s_1) = \varphi(s_2)$

e.g. s_1 and s_2 are different line segments but if we classify line segments by their lengths, they are regarded to be the "same".

Let S be an abstract space and let $\mathbb{P}$ be the set of all line segment in S.

Instead of using a length function, we can pick a subset (relation) $R \subseteq \mathbb{P} \times \mathbb{P}$ and say $s_1, s_2 \in \mathbb{P}$ are the "same" (or equivalent) if $(s_1, s_2) \in R$

Naturally, R should be chosen such that it is an equivalence relation

Example 2.2.7

Define a relation R on $\mathbb{Z}\times\mathbb{Z}^*$ (i.e. $R\subseteq(\mathbb{Z}\times\mathbb{Z}^*)\times(\mathbb{Z}\times\mathbb{Z}^*)$) where $\mathbb{Z}^*=\mathbb{Z}\setminus\{0\}$ as the following:

$(m,n)\sim(p,q)$ if $mq-np=0$.

(Think: If we have two fractions $\frac{m}{n}$ and $\frac{p}{q}$ where $m,p\in\mathbb{Z}$ and $n,q\in\mathbb{Z}^*$, they can be regarded as elements of $\mathbb{Z}\times\mathbb{Z}^*$. Also, they are the same if and only if $mq-np=0$.)

Show that R is an equivalence relation.

1) If $(m,n)\in R$, then $(m,n)\sim(m,n)$ since $mn-mn=0$

2) If $(m,n),(p,q)\in R$ and $(m,n)\sim(p,q)$, then $mq-np=0$ which means $pn-qm=0$ as well.

$\therefore (p,q)\sim(m,n)$

3) If $(m,n),(p,q),(r,s)\in R$, $(m,n)\sim(p,q)$ and $(p,q)\sim(r,s)$ then $mq-np=ps-qr=0$.

$$q(ms-nr)=msq-nps-nrq+nps$$
$$=s(mq-np)+n(ps-qr)=0$$

$q\neq 0 \Rightarrow ms-nr=0$

$\therefore (m,n)\sim(r,s)$

Exercise 2.2.2

Prove that $\sim$ is an equivalence relation on A if and only if both of the following hold:

1) $a\sim a \;\; \forall a\in A$

2) If $a\sim b$ and $a\sim c$, then $b\sim c$.

Remark: Symmetric and transitive conditions are put into a single statement.

Definition 2.2.3

Let $\sim$ be an equivalence relation on the set A.

$[a]=\{b\in A: a\sim b\}$ is called the equivalence class of a by $\sim$.

Any element of an equivalence class is called a representative.

$A/\sim=\{[a]: a\in A\}$ is called the quotient set of A by $\sim$.

Example 2.2.8

If R is the equivalence relation on $\mathbb{Z}$ which is given by $(a,b)\in R$ if $b-a$ is divisible by 3.

Note that $\cdots=[0]=[3]=[6]=\cdots$ $(=\{3m : m\in\mathbb{Z}\})$

$\cdots=[1]=[4]=[7]=\cdots$ $(=\{3m+1 : m\in\mathbb{Z}\})$

$\cdots=[2]=[5]=[8]=\cdots$ $(=\{3m+2 : m\in\mathbb{Z}\})$

$\mathbb{Z}_3=\mathbb{Z}/\sim=\{[0],[1],[2]\}$

There are only three equivalence classes and also we can observe that $\mathbb{Z}=[0]\amalg[1]\amalg[2]$.

Exercise 2.2.3

Consider the relation $\cong$ on $\mathbb{P}$ in example 2.2.4, describe $\mathbb{P}/\cong$.

(Ans: $\mathbb{P}/\cong$ is a set of equivalence classes where each of them is a subset of line segments with same length.

Also, note that $\mathbb{P}$ is the disjoint union of equivalence classes.)

We can generalize the above as the following.

Proposition 2.2.1

Let $\sim$ be an equivalence relation on the set A. Then

1) $a\in[a]$ for all $a\in A$

2) $[a]=[a']$ if and only if $a\sim a'$

3) A equals to the disjoint union of equivalence classes

proof:

1) Trivial, since $a\sim a$ for all $a\in A$.

2) "$\Rightarrow$" Assume $[a]=[a']$

From 1, $a'\in[a']=[a]$, so $a\sim a'$

"$\Leftarrow$" Assume $a\sim a'$.

Let $b\in[a']$. By definition $a'\sim b$.

$a\sim a'$ and $a'\sim b \Rightarrow a\sim b \Rightarrow b\in[a] \Rightarrow [a']\subseteq[a]$

By similar argument, we can show that $[a]\subseteq[a']$.

$\therefore [a]=[a']$.

3) Since every equivalence class is a subset of A, so does the union of equivalence classes.

For all $a \in A$, by 1, $a \in [a]$, so a belongs to the union of equivalence classes.

$\therefore$ union of equivalence classes $= A$ and what remains to show is the union is a disjoint union.

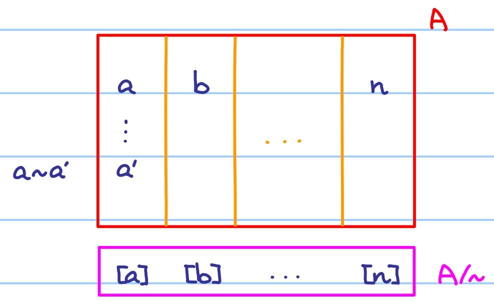
It is equivalent to show if $c \in [a] \cap [b]$, then $[a] = [b]$

$c \in [a] \cap [b] \Rightarrow a \sim c$ and $b \sim c$

$\Rightarrow a \sim b \quad (\because b \sim c \Rightarrow c \sim b)$

$\Rightarrow [a] = [b] \quad$ (by 2)

Sometimes, we say that the equivalence classes form a partition of A.



Defintion 2.2.4

A partial order relation on a set A is a relation $\preceq$ (usually denoted by $\leq$ instead of $\sim$) such that

1) (reflexive) $a \leq a$ for all $a \in A$

2) (antisymmetric) if $a \leq b$ and $b \leq a$, then $a = b$.

3) (transitive) if $a \leq b$ and $b \leq c$, then $a \leq c$.

Example 2.2.9 / Exercise 2.2.4

The relation in example 2.2.5 and $\leq$ in example 2.2.4 are actually partial order relations

Remark: A partial order relation on a set A tells us comparison of particular pairs of elements in A in some sense.

However, not every pair of elements in A can be compared, i.e. there exist distinct $a, b \in A$ such that both $a \leq b$ and $b \leq a$ are not true.

For example, consider "$|$" in example 2.2.5, we have $3 | 6$, but both $5 | 7$ and $7 | 5$ are not true

It leads the following definition

Definition 2.2.5

A total order relation on a set A is a relation $\preceq$ (usually denoted by $\leq$ instead of $\sim$) such that

1) (antisymmetric) if $a \leq b$ and $b \leq a$, then $a = b$.

2) (transitive) if $a \leq b$ and $b \leq c$, then $a \leq c$.

3) (totality) for all $a, b \in A$, $a \leq b$ or $b \leq a$.

Remark: The last condition guarantees us every pair of elements in A can be compared.

Furthermore, a strict total order relation $<$ associated by a total order relation $\leq$ can be defined by $a < b$ if and only if $a \leq b$ and $a \neq b$

Exercise 2.2.3

Show that a total order relation is a partial order relation.

(Ans: Consider (3) and $a = b \Rightarrow a \leq a \ \forall a \in A$)

Example 2.2.9 / Exercise 2.2.4

The relation $\leq$ in example 2.2.4 is a total order relation but the relation in example 2.2.8 is not.

Proposition 2.2.2

Let $<$ be a strict total order relation on A and let $a, b, c \in A$. Then,

1) (transitive) $a < b$ and $b < c \Rightarrow a < c$

2) (trichotomous) exactly one of $a < b$, $a = b$, $b < a$ is true

proof:

1) $a < b \Rightarrow a \leq b$ and $a \neq b$ $\qquad$ $b < c \Rightarrow b \leq c$ and $b \neq c$

$\therefore$ $a \leq b$ and $b \leq c \Rightarrow a \leq c$

What we remain to show is $a \neq c$.

Suppose that $a = c$, then we have $a \leq b$ and $b \leq c = a$ which forces $a = b$ (Contradiction)

$\therefore$ $a \neq c$

2) Direct consequence of totality of a total order relation.

Again, let S be an abstract space and let $\mathbb{P}$ be the set of all line segment in S

Instead of using a length function, we can pick a subset (relation) $R \subseteq \mathbb{P}/_{\cong} \times \mathbb{P}/_{\cong}$ and compare $[s_1],[s_2] \in \mathbb{P}/_{\cong}$ if $(s_1, s_2) \in R$.

Naturally, R should be chosen such that it is a total order relation.

2.3 Functions

Definition 2.3.1

A function f from A to B is a relation from A to B (i.e. $f \subseteq A \times B$) such that

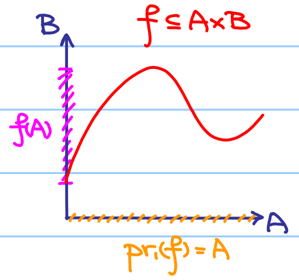
1) $pr_1(f) := \{a \in A : (a,b) \in f\} = A$

2) If $(a,b_1),(a,b_2) \in f$, then $b_1 = b_2$.

The sets A and B are said to be the domain and codomain of the function f respectively.

$(\text{range}(f) = f(A) =)$ $pr_2(f) := \{b \in B : (a,b) \in f\}$ is said to be the range of f.

We denote it by $f: A \to B$ and we write $f(a) = b$ or $a \xrightarrow{f} b$ if $(a,b) \in f$.



Remark: (1) guarantees that $f(a)$ is well-defined and

(2) guarantees that $a \in A$ is sent to a unique element in B

Example 2.3.1

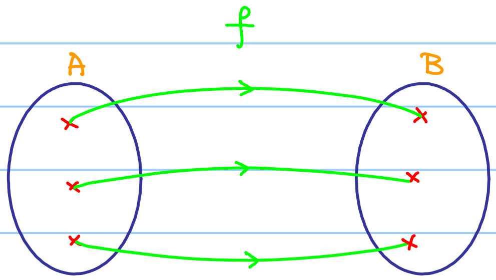
Addition of real numbers can also be regarded as a function $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $f(a,b) = a+b$.

In general, let S be a set, a function $f: S \times S \longrightarrow S$ is said to be a binary operation on A

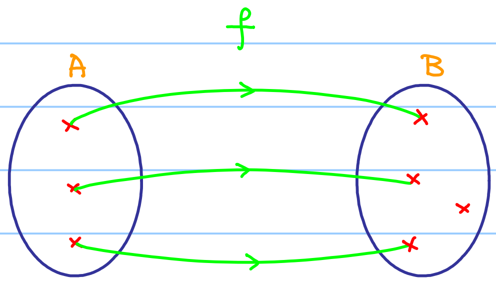
Sometimes, we simply write $a * b$ to denote $f(a,b)$.

Injective and Surjective Functions

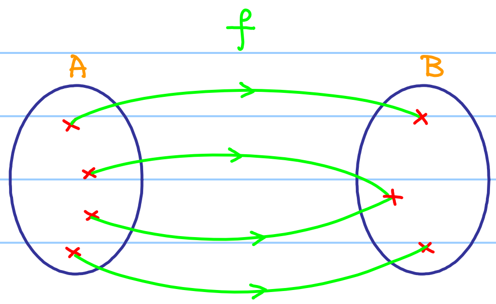
Intuitive idea :



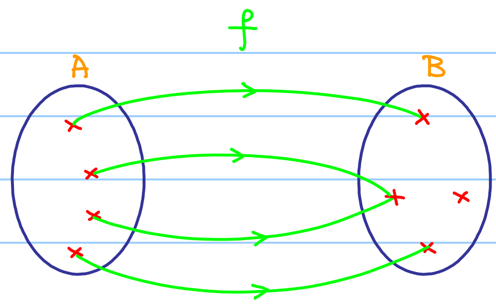
injective + surjective



injective but NOT surjective



surjective but NOT injective



injective : every $y \in \text{range}(f)$ comes from exactly one $x \in A$

surjective : every $y \in B$ comes from one $x \in A$

Definition 2.3.1

Let $f : A \to B$ be a function.

1) f is said to be an injective function if

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

(Explanation : Once the output are the same , the inputs must be the same !)

2) f is said to be a surjective function if

$$\forall y \in B , \exists x \in A \text{ s.t. } f(x) = y \qquad (f(A) = B)$$

If f is both injective and surjective , then it is said to be bijective .

Definition 2.3.2

Let $f : A \to B$ be a function. If $g : B \to A$ is a function such that

1) $g(f(x)) = x \quad \forall x \in A$

2) $f(g(y)) = y \quad \forall y \in B$

Then g is said to be an inverse of f.

Theorem 2.3.1

1) If an inverse of f exists, it is unique, so we denote it by f^{-1}.

2) f has an inverse if and only if f is bijective.

Example 2.3.2

$f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is neither injective nor surjective.

$f : [0,\infty) \to [0,\infty)$ defined by $f(x) = x^2$ is bijective. Its inverse $f^{-1} : [0,\infty) \to [0,\infty)$ is denoted by $f^{-1}(x) = \sqrt{x}$.

Example 2.3.3

Let $A = \{1,2,3\}$, $B = \{a,b,c\}$ and let $f : A \to B$ defined by $f(1) = a$, $f(2) = b$, $f(3) = c$.

It can be check directly that f is bijective.

Remark: Naively, if $f : A \to B$ is a bijective function, then the "number" of elements in A and B are the same.

Example 2.3.1

Let $\mathbb{P}$ be the set of line segment in $\mathbb{R}^2$ (in usual sense)

and let $\varphi : \mathbb{P} \to \mathbb{R}^+$ be the distance function, i.e $\varphi(s)$ = distance of line segment s.

a) Show that φ is surjective but not injective.

b) Let $\cong$ be the equivalence relation on $\mathbb{P}$ such that $s_1 \cong s_2$ if $\varphi(s_1) = \varphi(s_2)$.

Let $\tilde{\varphi} : \mathbb{P}/_{\cong} \to \mathbb{R}^+$ be a function defined by $\tilde{\varphi}([s]) = \varphi(s)$.

Show that $\tilde{\varphi}$ is well-defined and bijective.

(Hence, $\mathbb{P}/_{\cong}$ has the same "number" of elements as $\mathbb{R}$)

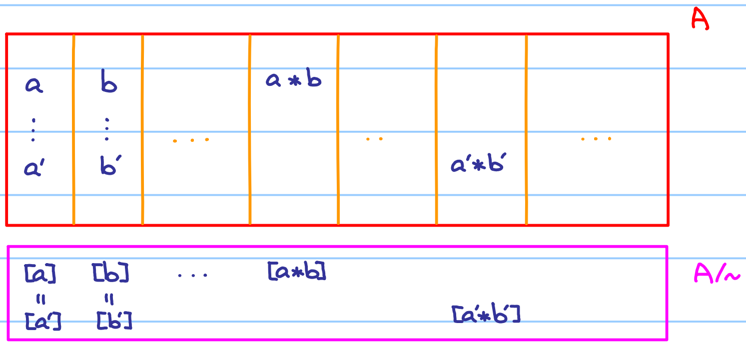
2.4 More on Equivalence Relation

Suppose $\sim$ is an equivalence relation on A and $*$ is a binary operation on A.

Main question: Does $*$ induce a binary operation $\tilde{*}$ on $A/\sim$?

Naturally: We try to define $[a]\,\tilde{*}\,[b] = [a*b]$.

Trouble: It may happen that $a' \in [a]$, $b' \in [b]$ (i.e. $a \sim a'$ and $b \sim b'$) but $[a'*b'] \neq [a*b]$ (i.e. $a*b \not\sim a'*b'$).



What we require: If $a \sim a'$, $b \sim b'$, then $a*b \sim a'*b'$.

Theorem 2.4.1

$*$ induces a binary operation $\tilde{*}$ on $A/\sim$ iff the above condition holds.

For simplicity, we abuse the use of notation and denote the binary operation on $A/\sim$ by $*$ again.

Example 2.4.1

Define a relation R on $\mathbb{Z}\times\mathbb{Z}^*$ as in example 2.2.5.

Define a binary operation (addition $+$) on $\mathbb{Z}\times\mathbb{Z}^*$ by $(m,n)+(p,q) = (mq+np, nq)$.

addition defined on $\mathbb{Z}\times\mathbb{Z}^*$

ordinary addition on $\mathbb{Z}$

(Think: Regard (m,n) as $\frac{m}{n}$, $(m,n)+(p,q)$ is defined as $\frac{mq+np}{nq}$)

If $(m,n)\sim(m',n')$ and $(p,q)\sim(p',q')$, i.e. $mn'-nm' = pq'-qp' = 0$

$(m',n')+(p',q') = (m'q'+n'p', n'q')$.

Then $(mq+np)n'q' - nq(m'q'+n'p') = 0 \Rightarrow (m,n)+(p,q) \sim (m',n')+(p',q')$

$\therefore$ We can define addition on $\mathbb{Q} = \mathbb{Z}\times\mathbb{Z}^*/\sim$

Remark: Usually, we say $\frac{1}{2}, \frac{3}{4} \in \mathbb{Q}$. To be precise, it should be $[\frac{1}{2}], [\frac{3}{4}] \in \mathbb{Q}$

$[\frac{1}{2}] + [\frac{3}{4}] := [\frac{1}{2} + \frac{3}{4}]$ ($+$ is defined on $\mathbb{Q}$, $+$ is defined on $\mathbb{Z}\times\mathbb{Z}^*$)

$= [\frac{1\times4+3\times2}{2\times4}] = [\frac{10}{8}] = [\frac{5}{4}]$ ($\because \frac{10}{8} \sim \frac{5}{4}$)

However, we can freely take other representatives in $[\frac{1}{2}], [\frac{3}{4}]$, say $\frac{3}{6} \in [\frac{1}{2}]$ and $\frac{9}{12} \in [\frac{3}{4}]$ and

$[\frac{1}{2}] + [\frac{3}{4}] := [\frac{3}{6} + \frac{9}{12}] = [\frac{3\times12+9\times6}{6\times12}] = [\frac{90}{72}] = [\frac{5}{4}]$

Exercise 2.4.1

Let $\sim$ be the equivalence relation on $\mathbb{Z}$ defined in example 2.2.2.

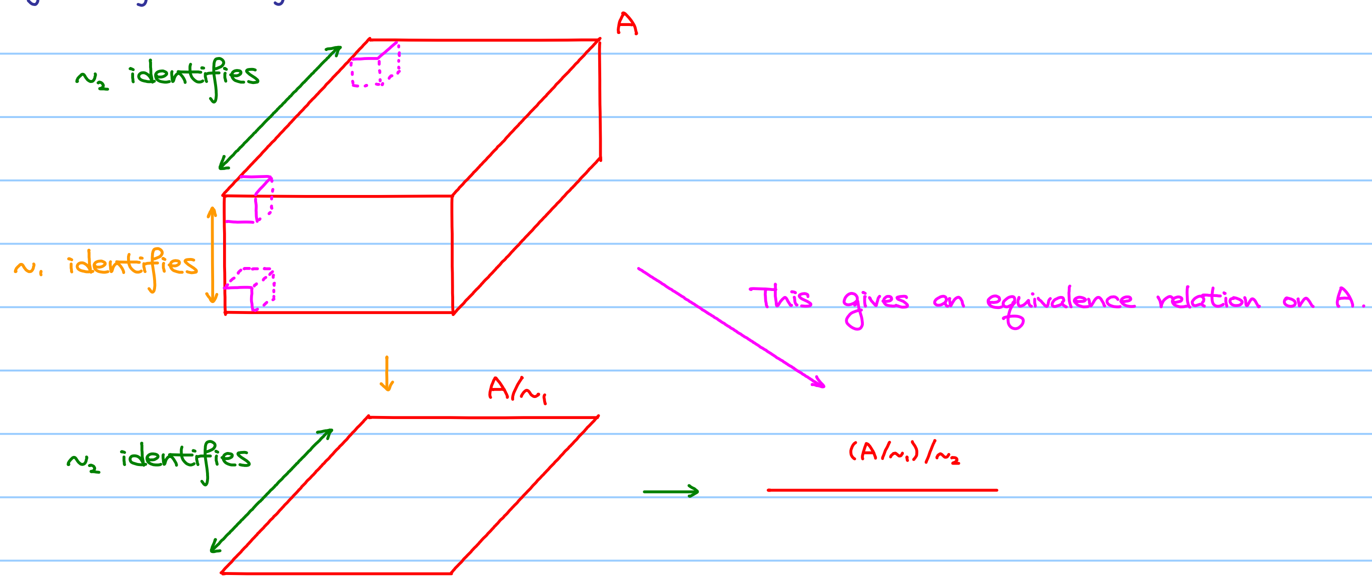
Prove that addition of $\mathbb{Z}$ induces an addition on $\mathbb{Z}/\sim$.

Exercise 2.4.2

Let $\sim_1$ be an equivalence relation on A and $\sim_2$ be an equivalence relation on $A/\sim_1$

Let $\sim$ be an relation on A defined by $a\sim b$ if $[a]\sim_2[b]$.

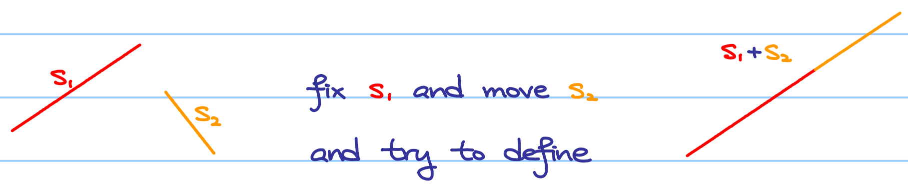
Prove that $\sim$ is an equivalence relation on A and $A/\sim = (A/\sim_1)/\sim_2$, to be precise there exists a bijective function $f: A/\sim \to (A/\sim_1)/\sim_2$.



Think:

Let $\mathbb{P}$ be the set of line segment in $\mathbb{R}^2$ (in usual sense).

How do we define addition on $\mathbb{P}$?



However, there is an ambiguity, which endpoint of s_1 should we connect?

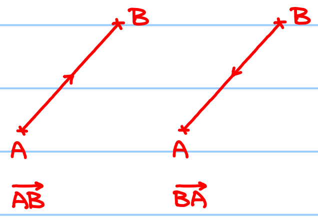
Therefore, instead of $\mathbb{P}$, actually we define addition on $\mathbb{P}/\cong$. Let $[s_1],[s_2] \in \mathbb{P}/\cong$.

Then, we define $[s_1]+[s_2] = [s]$ where $\varphi(s) = \varphi(s_1)+\varphi(s_2)$ (Why it is well-defined?)

However, if S is an abstract space and there is no distance function anymore but only an equivalence relation $\cong$ on $\mathbb{P}$, how do we define addition on $\mathbb{P}/\cong$?

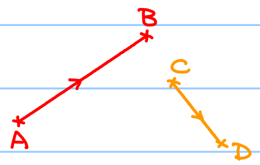
Here is the idea (detail will be discussed later):

Let $\widetilde{\mathbb{P}}$ be the set of all oriented line segments.

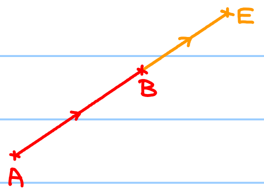


they are regarded as distinct elements in $\widetilde{\mathbb{P}}$

Addition on $\widetilde{\mathbb{P}}$ can be defined as :



try to define $\overrightarrow{AB} + \overrightarrow{CD}$ as



where E lies on the line passing through A and B

Define a relation $\sim$ on $\widetilde{\mathbb{P}}$ such that $\overrightarrow{AB} \sim \overrightarrow{BA}$

Actually, $\sim$ is an equivalence relation and $(\widetilde{\mathbb{P}}/\sim) = \mathbb{P}$.

Then it gives an equivalence relation $\sim_0$ on $\widetilde{\mathbb{P}}$ such that $\widetilde{\mathbb{P}}/\sim_0 = (\widetilde{\mathbb{P}}/\sim)/\cong = \mathbb{P}/\cong$.

We would also like to show that addition on $\widetilde{\mathbb{P}}$ induces an addition on $\widetilde{\mathbb{P}}/\sim_0 = \mathbb{P}/\cong$.

Recall theorem 2.4.1, what we have to show is :

if $\overrightarrow{AB} \sim_0 \overrightarrow{A'B'}$ and $\overrightarrow{CD} \sim_0 \overrightarrow{C'D'}$, then $\overrightarrow{AB} + \overrightarrow{CD} \sim_0 \overrightarrow{A'B'} + \overrightarrow{C'D'}$.